Electromagnetism: Electrostatics (Solutions)

FIZIKA SPhO Training

July 2025

Contents

1 Notes

In electrostatics, we analyse the forces and motion of static charges. Hereby, we shall take $k = \frac{1}{4\pi\varepsilon_0} \approx 8.99 \times 10^9\,\mathrm{N\,m^2/C^2}$.

1.1 The Basic Quantities

In electrostatics, the four basic quantities you will deal with are:

1. Electric Force, $\mathbf{F}_E$
2. Electric Field, $\mathbf{E}$
3. Electric Potential Energy, U_E
4. Electric Potential, V_E

Notice that these quantities are *very similar* to what you've seen in gravitation!

1.1.1 Electric Force and Electric Field

Consider two point charges q_1 and q_2 located at position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Let the separation vector be $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

The electric forces by each point charge on the other charge are given by

$$\mathbf{F}_{E,1\,by\,2} = -\frac{kq_1q_2}{|\mathbf{r}|^3}\mathbf{r} \tag{1}$$

$$\mathbf{F}_{E,2\,by\,1} = \frac{kq_1q_2}{|\mathbf{r}|^3}\mathbf{r} \tag{2}$$

The electric field can be thought of as the electric force per unit charge. Let the charge Q be placed at the origin. The electric field at a position vector $\mathbf{r}$ away is given by

$$\mathbf{E} = \frac{kQ}{|\mathbf{r}|^3}\mathbf{r} \tag{3}$$

Remark. The signs of the charges in Equations (1) to (3) are important!

1.1.2 Electric Potential Energy and Electric Potential

Consider two point charges again. The electric potential energy is given by

$$U_E = \frac{kq_1q_2}{|\mathbf{r}|} \tag{4}$$

The electric potential can be thought of as the electric potential energy per unit charge. Thus, the electric potential at a position vector $\mathbf{r}$ away is given by

$$V_E = \frac{kQ}{|\mathbf{r}|} \tag{5}$$

Notice that Equations (1) to (5) look *very similar* to Equations (1) to (5) in the gravitation handout! We can draw an analogy between electrostatics and gravitation:

$$k \longleftrightarrow G,\ q \longleftrightarrow m \tag{6}$$

which means any electrostatics problem can be solved as a gravitation one, and vice-versa!

1.1.3 Relationships Between Quantities

You should expect similar relationships to hold, as in gravitation:

$$F_E = -\frac{dU_E}{dr} \tag{7}$$

$$U_E = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_E \cdot d\mathbf{r} \tag{8}$$

$$E = -\frac{dV_E}{dr} \tag{9}$$

$$V_E = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r} \tag{10}$$

The "square" between the four quantities you saw in the gravitation handout can also be used.

1.2 Continuous Charge Distributions

We have been talking about discrete charge distributions, but what about continuous charge distributions? Continuous charge distributions can have either linear charge density λ, surface charge density σ, or volume charge density ρ.

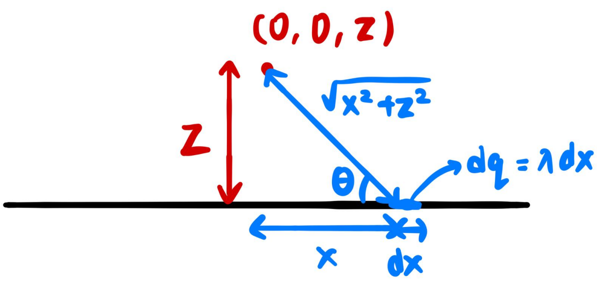
1.2.1 Naive Integration

We are usually interested in finding the electric field or electric potential. One way to attack such problems is through naive integration, as the example below illustrates.

Example 1.1. Find the electric field at a distance z above the midpoint of a straight wire of length $2L$ that carries a uniform linear charge density λ.

Let the wire lie along the x-axis with its centre at the origin. Clearly, due to symmetry, only the z-component of the electric field survives upon integration across the wire.

Consider an infinitesimal charge $dq = \lambda\, dx$ in the figure below:



The infinitesimal electric field produced by this infinitesimal charge is

$$dE = \frac{k\lambda\, dx}{r^2} = \frac{k\lambda\, dx}{x^2 + z^2}$$

The net electric field is the sum of all the z-components, hence

$$E_{net} = \int dE \sin\theta = \int_{-L}^{L} \frac{k\lambda\, dx}{x^2 + z^2} \frac{z}{\sqrt{x^2 + z^2}} = k\lambda z \int_{-L}^{L} \frac{dx}{(x^2 + z^2)^{\frac{3}{2}}} = \frac{2k\lambda L}{z\sqrt{L^2 + z^2}}$$

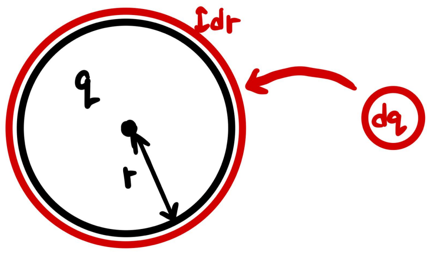
Just like in gravitation, brute force integration will become very involved for more complicated geometries. There are usually better ways to compute fields and potentials for continuous charge distributions.

1.2.2 "Building Up Layers"

The idea of "building up layers" is very important for potential energy. Let's consider the example below.

Example 1.2. Find the electric potential energy of a spherically symmetric charge distribution of total charge Q and radius R.

We can imagine constructing this spherically symmetric charge distribution by slowly adding on the layers of the sphere:



Let the volume charge density be $\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$. The infinitesimal potential energy between the new layer and the existing sphere is

$$dU = \frac{kq\,dq}{r} = \frac{k\left(\frac{4}{3}\rho\pi r^3\right)\,d\left(\frac{4}{3}\rho\pi r^3\right)}{r} = \frac{4}{3}k\pi\rho r^2\left(4\pi\rho r^2\,dr\right) = \frac{16}{3}k\pi^2\rho^2 r^4\,dr$$

Hence, we can integrate from 0 to R to find the total potential energy:

$$U = \int dU = \int_0^R \frac{16}{3}k\pi^2\rho^2 r^4\,dr = \frac{16}{3}k\pi^2\rho^2\left(\frac{R^5}{5}\right) = \frac{3kQ^2}{5R}$$

1.3 Electric Dipoles

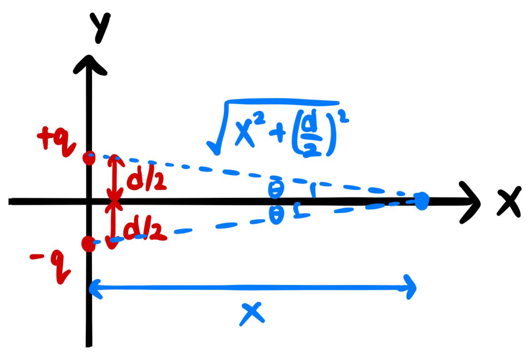
An electric dipole consists of a positive point charge $+q$ and a negative point charge $-q$ separated by a **small** distance d.

1.3.1 Electric Field of a Dipole

To calculate this, we treat the dipole as two point charges.

Example 1.3. Consider an electric dipole with a positive charge $+q$ located at $\left(0, \frac{d}{2}\right)$ and a negative charge $-q$ located at $\left(0, -\frac{d}{2}\right)$. (i) Find the electric field at $(x, 0)$. (ii) Find the electric field at $(0, y)$. Assume $x, y >> d$.

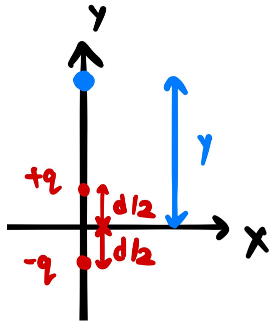
(i) The diagram of the set-up is as such:



Clearly, the x-component cancels out due to symmetry. Thus,

$$E_{net} = 2E_y = 2E\sin\theta = 2\left(\frac{kq}{x^2+\left(\frac{d}{2}\right)^2}\right)\left(\frac{\frac{d}{2}}{\sqrt{x^2+\left(\frac{d}{2}\right)^2}}\right) = \frac{kqd}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}} \approx \frac{kqd}{x^3}$$

(ii) The diagram of the set-up is as such:



Clearly, there is no x-component. Thus, defining upwards as positive,

$$E_{net} = E_+ + E_- = \frac{kq}{\left(y-\frac{d}{2}\right)^2} - \frac{kq}{\left(y+\frac{d}{2}\right)^2} = kq\left(\frac{\left(y+\frac{d}{2}\right)^2-\left(y-\frac{d}{2}\right)^2}{\left(y^2-\left(\frac{d}{2}\right)^2\right)^2}\right) \approx kq\left(\frac{2yd}{y^4}\right) = \frac{2kqd}{y^3}$$

Interestingly, they "differ" by a factor of 2.

1.3.2 Electric Dipole Moment

Electric dipoles are often characterised by their **electric dipole moment, p**, defined as

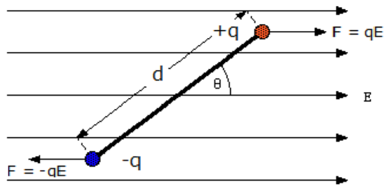
$$\mathbf{p} = q\mathbf{d} \tag{11}$$

where **p** and **d** are defined to point **from the negative to the positive charge.**

You often see the product qd pop up when dealing with dipoles (you already saw it in Example 1.3). The electric dipole moment is just a convenient quantity to simplify the product.

1.3.3 Potential Energy and Torque of an Electric Dipole

When an electric dipole is placed into an **external electric field, E**, each charge experience a force. Due to the finite size of the dipole, this leads to a torque.



This torque is given by

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} \tag{12}$$

There is also a potential energy associated with this configuration, given by

$$U = -\mathbf{p}\cdot\mathbf{E} \tag{13}$$

1.4 Electric Flux and Gauss' Law

In gravitation, you have seen how to calculate gravitational flux and apply Gauss' Law. Given the similarities between electrostatics and gravitation, there similarly exists **electric flux,** Φ_E, and **Gauss' Law for Electricity.**

Electric flux is defined by:

$$\Phi_E = \oiint_A \mathbf{E} \cdot d\mathbf{A} \tag{14}$$

while Gauss' Law for Electricity states:

$$\oiint_A \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enclosed}}{\varepsilon_0} \tag{15}$$

Here, $q_{enclosed}$ refers to the **total** charge enclosed by the Gaussian surface. **While the area integral looks daunting, in most cases symmetry allows you to just multiple field by area directly with minimal/zero integration!**

Remark. You need to be careful when calculating $q_{enclosed}$, as charges can be positive or negative (unlike masses, which are only positive)! Make sure to account for the correct signs.

From gravitation, you already know the different types of symmetries to spot when applying Gauss' Law. Let's go through them again in the context of electrostatics.

1.4.1 Spherical Symmetry

Example 1.4. Consider a sphere with uniform charge density ρ and radius R. Using Gauss' Law, find the electric field at a distance r away from it.

By symmetry the electric field must be radially outward and have the same magnitude everywhere on any concentric spherical surface, thus we choose a sphere of radius r concentric with the charge distribution to be our Gaussian surface.
Computing the charge enclosed,

$$q_{\text{enc}} = \rho \left(\frac{4}{3}\pi r^3\right).$$

Applying Gauss' law,

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A} = E(r)\,(4\pi r^2) = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\varepsilon_0}.$$

Solving for $E(r)$**,**

$$E(r) = \frac{\rho\, r}{3\varepsilon_0}, \qquad 0 \le r < R.$$

Hence the electric-field magnitude is *directly proportional to* r—i.e. linear—inside the sphere. The field points radially outward (or inward if $\rho < 0$) and satisfies:

$$E(0) = 0, \quad E(R) = \frac{\rho R}{3\varepsilon_0}.$$

For $r \ge R$ all the charge $Q = \rho\frac{4}{3}\pi R^3$ is enclosed, so

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{\rho R^3}{3\varepsilon_0}\frac{1}{r^2},$$

which joins smoothly to the interior result at $r = R$.

1.4.2 Cylindrical Symmetry

Example 1.5. Consider an infinitely long charged wire of linear charge density λ. Using Gauss' Law, find the electric field at a distance x away from it.

Clearly, with the cylindrical symmetry around the wire, we draw a Gaussian cylinder of radius x and some length L around the wire. (Revisit gravitation if you aren't sure why.)

Thus, by Gauss' Law,

$$E\,(2\pi x L) = \frac{\lambda L}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 x} = \frac{2k\lambda}{x}$$

1.4.3 Planar Symmetry

Example 1.6. Consider an infinite charged plane with area charge density σ. Using Gauss' Law, find the electric field at a distance x away from it.

Clearly, with the planar symmetry around the plane, we draw a Gaussian "pillbox" of cross-sectional area A, with its faces at $\pm x$ parallel to the plane. (Revisit gravitation if you aren't sure why.)

Thus, by Gauss' Law,

$$E\,(2A) = \frac{\sigma A}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0}$$

1.4.4 Charged Boundaries

When we cross a boundary filled with charges, how does the electric field change right below and above the charged boundary?

Consider any random boundary filled with charge density σ. We can draw a Gaussian "pillbox", and by Gauss' Law,

$$E_{top}A - E_{bottom}A = \frac{\sigma A}{\varepsilon_0} \tag{16}$$

Thus we have:

$$\Delta \mathbf{E}_\perp = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}} \tag{17}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the plane.

This implies that electric field is **discontinuous** across charged boundaries!

A charged boundary also gives rise to **electrostatic pressure,** P_E, which is the electric force per unit area on the boundary:

$$P_E = \frac{\sigma^2}{2\varepsilon_0} \tag{18}$$

which is very useful when calculating forces.

Interestingly, this may be a little unintuitive, but P_E is same as the **electric potential energy density** (by volume), u_E! (You can perform a quick sanity check by dimensional analysis). u_E is defined as

$$u_E = \frac{U}{V} = \frac{1}{2}\varepsilon_0 E^2 \tag{19}$$

Integrating over the volume occupied by the electric field gives the total electric potential energy:

$$U = \int u_E \, dV = \int \frac{1}{2}\varepsilon_0 E^2 \, dV$$

1.5 Conductors

Conductors are a special type of material in which charges will redistribute themselves such that a lowest energy configuration is achieved. Most commonly, you would think of metals as conductors.

1.5.1 Properties of Conductors

There are a few general properties of conductors:

1. Inside a conductor, the net electric field $\mathbf{E} = \mathbf{0}$ (including external and induced fields).
2. The charge density $\rho = 0$ inside; excess charge resides only on the surface.
3. Just outside the conductor, $\mathbf{E}$ is normal to the surface.
4. The conductor's surface is equipotential. Since $\mathbf{E} = \mathbf{0}$ inside, the potential V is uniform throughout.

These properties are sufficient to determine the charge distribution on conductors.

We can also discuss the electrostatic pressure here. Since we know that electric field inside the conductor is zero, the pressure can be expressed as such:

$$P_{E,\, conductor} = \sigma \left(\frac{E_{out} + 0}{2} \right) = \frac{\sigma E_{out}}{2} \tag{20}$$

1.5.2 Grounding

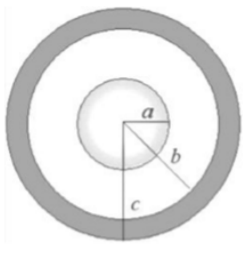
Grounding is the process of removing *excess* charges on an object by transferring electrons between it and the "ground" (an infinite reservoir of electrons).

In essence, grounding sets the **potential** of the object to be 0 (the reference).

Remark. It is **wrong** to assume that the total charge of the object is 0 after grounding! Grounding only removes *excess* charges to the point where the potential becomes 0.

In problems involving conductors/grounded conductors, you need to determine the charges on each **surface.** The example below illustrates.

Example 1.7 (Ricardo)**.** Two concentric spherical conductors are described in the figure below. The smaller one is a solid sphere of radius a, and is charged with $+Q$. The larger one is a hollow sphere of inner radius b and outer radius c, and is charged with $-3Q$. (i) Find the functions $E(r)$ and $V(r)$. (ii) The larger sphere is now grounded. Find the functions $E(r)$ and $V(r)$. (iii) The larger sphere is now disconnected from the ground and is returned to having a charge $-3Q$. The smaller sphere is now grounded. Find the functions $E(r)$ and $V(r)$.



(i) Notice that the $-3Q$ on the larger sphere must spread out to have $-Q$ on the inner surface and $-2Q$ on the outer surface. This configuration ensures that when we consider the region

$b < r < c$, the enclosed charge is 0, hence the field is 0 inside the conductor.

Following which, this is just a simple application of Gauss' Law for spheres:

1. When $r < a$, there is no charge in the interior of the conductor, so $E = 0$.
2. When $a < r < b$, the enclosed charge is $q_{enclosed} = Q$, so $E = \frac{kQ}{r^2}$.
3. When $b < r < c$, we established that $E = 0$.
4. When $r > c$, the enclosed charge is $q_{enclosed} = -2Q$, so $E = -\frac{2kQ}{r^2}$.

Thus, the electric field is

$$E(r) = \begin{cases} 0, & r < a \\ \frac{kQ}{r^2}, & a < r < b \\ 0, & b < r < c \\ -\frac{2kQ}{r^2} & r > c \end{cases}$$

We can begin integrating to find the potential, and you should obtain

$$V(r) = \begin{cases} -\frac{2kQ}{c} - kQ\left(\frac{1}{a} - \frac{1}{b}\right), & r < a \\ -\frac{2kQ}{c} - kQ\left(\frac{1}{r} - \frac{1}{b}\right), & a < r < b \\ -\frac{2kQ}{c}, & b < r < c \\ -\frac{2kQ}{r}, & r > c \end{cases}$$

The potential inside the conductor regions is constant, as you'd expect.

(ii) When the larger sphere is grounded, it has 0 potential. The only configuration that achieves this is if it has no charge (and hence the same potential as infinity, which has 0 potential).

Applying Gauss' Law, you should obtain

$$E(r) = \begin{cases} 0, & r < a \\ \frac{kQ}{r^2}, & a < r < b \\ 0, & b < r < c \\ 0, & r > c \end{cases}$$

We can begin integrating to find the potential, and you should obtain

$$V(r) = \begin{cases} kQ\left(\frac{1}{b} - \frac{1}{a}\right), & r < a \\ kQ\left(\frac{1}{b} - \frac{1}{r}\right), & a < r < b \\ 0, & r > b \end{cases}$$

(iii) It is very easy to fall into the trap and say that the charge on the smaller sphere is now 0. However, this is **wrong!** Clearly, the potential of the smaller sphere is not 0 in this case.

The correct method is to assume the smaller sphere, the inner surface of the larger sphere, and the outer surface of the larger sphere take on unknown charges q_1, q_2 and q_3 respectively.

Since the larger sphere is not connected to the ground, conservation of charge must be obeyed:

$$q_2 + q_3 = -3Q$$

For there to be no field inside $b < r < c$, we must have:

$$q_1 = -q_2$$

Now, we can express the potential at $r = a$ in terms of these unknown charges:

$$V(a) = k\left(\frac{q_1}{a} + \frac{q_2}{b} + \frac{q_3}{c}\right) = 0 \quad \Rightarrow \quad q_1\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) = \frac{3Q}{c}$$

With this, we can solve for all the unknown charges:

$$q_1 = \frac{3Q}{1 + c\left(\frac{1}{a} - \frac{1}{b}\right)}, \; q_2 = -\frac{3Q}{1 + c\left(\frac{1}{a} - \frac{1}{b}\right)}, \; q_3 = -\frac{3Qc\left(\frac{1}{a} - \frac{1}{b}\right)}{1 + c\left(\frac{1}{a} - \frac{1}{b}\right)}$$

Now, we can repeat a similar procedure to the other parts. Applying Gauss' Law, you should obtain

$$E(r) = \begin{cases} 0, & r < a \\ \frac{kq_1}{r^2}, & a < r < b \\ 0, & b < r < c \\ \frac{k(q_1 - 3Q)}{r^2}, & r > c \end{cases}$$

We can begin integrating to find the potential, and you should obtain

$$V(r) = \begin{cases} 0, & r > a \\ k\left(\frac{q_1}{a} - \frac{q_1}{b} + \frac{q_1 - 3Q}{c}\right), & a < r < b \\ k\left(\frac{q_1 - 3Q}{c}\right), & b < r < c \\ k\left(\frac{q_1 - 3Q}{r}\right), & r > c \end{cases}$$

1.6 Ideas

Many tricky electromagnetism problems involve the use of the following ideas.

1.6.1 Method of Images

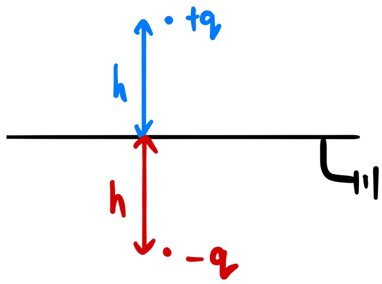
Often times, when dealing with charges and conductors, it may be difficult to calculate forces, fields and potentials. Let's take a look at a simple set-up below.

Example 1.8. Consider a charge $+q$ placed at a distance h above a thin, **grounded,** infinite conducting plane. (i) Find the force between the charge and the plane. (ii) Find the potential energy of this configuration.

(i) Naively, if you knew nothing about the method of images, you would perhaps to try find the induced charge distribution on the plane (setting the potential to be 0 everywhere on the conductor), and integrate across the plane. However, this is really tedious.

Instead, ask yourself: is there any other charge configuration that gives the same **boundary conditions?** (In this case, this refers to the potential of the whole plane being 0.)

Consider placing a charge $-q$ at a distance h below the plane.



Notice that by symmetry, *all* points on the plane have 0 potential (because the 2 charges cancel each other out)! This is the configuration that we are looking for.

With this, the force is simple:

$$F = \frac{kq^2}{(2h)^2} = \frac{kq^2}{4h^2}$$

(ii) Now that you know about this configuration, you may be tempted to just write

$$U = \frac{kq^2}{2h}$$

treating it as the potential energy between two point charges.

However, this is **wrong!** The reason is because the charge $-q$ is *not* a real charge. It is what we call an **image charge,** solely for the purpose of satisfying the boundary conditions.

The **correct answer** requires the definition of potential energy:

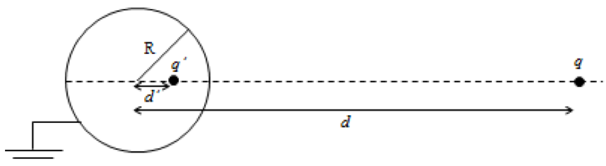
$$U = -\int_{\infty}^{h} F\,dh = -\int_{\infty}^{h} \frac{kq^2}{4h^2}\,dh = \frac{kq^2}{4h}$$

You might realise that the correct answer is exactly half of the wrong answer. The energy that we "over-counted" in the wrong answer comes from the interaction of the charges *on* the conducting plane (which are *real* charges!)

Remark. How are we so sure that this random configuration that we pulled out of nowhere is the *correct* configuration? The answer is the uniqueness theorem. Again, this is a case whereby you don't need to know how it works - you just need to know it exists and how to use it!

Other than the thin, grounded, infinite conducting plane, the other set-up you must recognise is the **hollow grounded sphere.**

Example 1.9 (SPhO 2015)**.** Consider a charge q a distance d away from a grounded spherical conductor of radius R. The electric field and potential outside the sphere can be represented by an image charge q' a distance d' away from the centre of the sphere (there is no need to prove this). (i) Find q' and d'. (ii) Find the force F between the conductor and charge q, and determine whether it is attractive or repulsive.



(i) Again, the boundary condition is that the potential on the sphere is 0. Set up coordinates such that the charges lie along the x-axis, and the sphere is centred at the origin.

The potential at $(R, 0)$ and $(-R, 0)$ must be 0. Thus,

$$\frac{kq}{d-R} + \frac{kq'}{R-d'} = 0, \quad \frac{kq'}{R+d'} + \frac{kq}{d+R} = 0$$

These two simultaneous equations can be solved for q' and d', to obtain

$$d' = \frac{R^2}{d}, \quad q' = -\frac{qR}{d}$$

If you know some geometry, you can see that the image charge q' and the real charge q are inversions of each other with respect to the sphere.

(ii) The force F can be found by considering the image charge:

$$F = \frac{kqq'}{(d-d')^2} = \frac{kq\left(-\frac{qR}{d}\right)}{\left(d-\frac{R^2}{d}\right)^2} = -\frac{kq^2R}{d\left(d-\frac{R^2}{d}\right)^2}$$

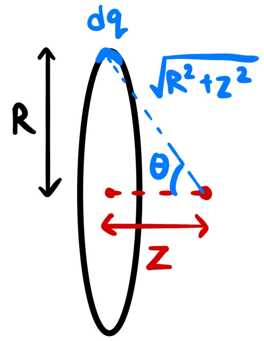
Clearly, the force is attractive as the charges have opposite signs.

1.6.2 Gauss' Law with $q_{enclosed} = 0$

So far, we have only applied Gauss' Law with $q_{enclosed}$ being non-zero. It may also seem weird and useless to apply Gauss' Law with $q_{enclosed} = 0$. However, when applied correctly, it can speed up calculations! The following example illustrates.

Example 1.10 (IPhO 2021, modified)**.** Consider a thin ring of linear charge density λ and radius R. (i) Find the electric field at a distance z away from the centre of the ring, **along its axis.** (ii) Find the electric field at a distance x away from the centre of the ring, **in its plane.** You may assume $x, z << R$.

(i) The first part is simple and is more easily done by naive integration.

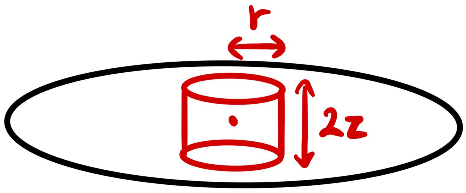


Only the z-component survives upon integration, hence

$$E = \int dE\cos\theta = \int \frac{k\,dq}{z^2+R^2}\frac{z}{\sqrt{z^2+R^2}} = \frac{kqz}{(z^2+R^2)^{\frac{3}{2}}} = \frac{2\pi k\lambda Rz}{(z^2+R^2)^{\frac{3}{2}}} \approx \frac{2\pi k\lambda z}{R^2}$$

Not bad for around half a point on the IPhO!

(ii) This is where we use the Gauss' Law trick, which requires the result of (i). Consider a small Gaussian cylinder of radius r and height $2z$ constructed at the centre of the ring:



Since $q_{enclosed} = 0$, by Gauss' Law,

$$\Phi_{E,\,total} = \oint\!\!\oint_{top} \mathbf{E}\cdot d\mathbf{A} + \oint\!\!\oint_{bottom} \mathbf{E}\cdot d\mathbf{A} + \oint\!\!\oint_{side} \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enclosed}}{\varepsilon_0} = 0$$

Because the cylinder is small, we can evaluate the flux through each of the faces by assuming a constant field across the surfaces. Thus,

$$\oint\!\!\oint_{top} \mathbf{E}\cdot d\mathbf{A} = \oint\!\!\oint_{bottom} \mathbf{E}\cdot d\mathbf{A} = \pi r^2 \left(\frac{2\pi k\lambda z}{R^2}\right) = \frac{2\pi^2 k\lambda z r^2}{R^2}$$

Thus,

$$\oint\!\!\oint_{side} \mathbf{E}\cdot d\mathbf{A} = -\left(\oint\!\!\oint_{top} \mathbf{E}\cdot d\mathbf{A} + \oint\!\!\oint_{bottom} \mathbf{E}\cdot d\mathbf{A}\right) = -\frac{4\pi^2 k\lambda z r^2}{R^2} = E\left(2\pi r\left(2z\right)\right)$$

Thus, the electric field along the axis of the ring is

$$E = -\frac{\pi k\lambda r}{R^2}$$

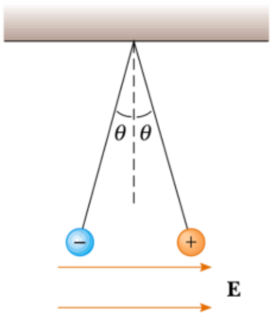
where the negative sign implies that the field points radially inwards.

By using this trick, we didn't even have to integrate at all!

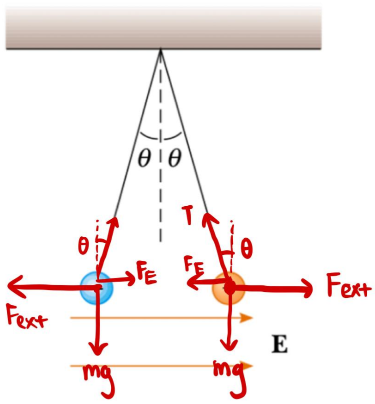
2 Problems

Problems are arranged in roughly increasing difficulty.

Problem 2.1 (SPhO 2008)**.** Two small spheres, each of mass 2.00 g, are suspended by light strings 10.0 cm in length. A uniform electric field is applied in the x-direction. The spheres have charges equal to -5.00×10^{-8} C and $+5.00 \times 10^{-8}$ C. Determine the electric field that enables the spheres to be in equilibrium at an angle 10.0°.



Solution. We first draw the FBD of both spheres:



Balancing forces in the horizontal and vertical directions,

$$F_E + T\sin\theta = F_{ext},\ \ mg = T\cos\theta$$

We can evaluate F_E and F_{ext}:

$$F_E = \frac{kq^2}{(2L\sin\theta)^2} = \frac{kq^2}{4L^2\sin^2\theta},\ \ F_{ext} = qE$$

Solving these equations for E, you should obtain

$$E = \frac{1}{q}\left(\frac{kq^2}{4L^2\sin^2\theta} + mg\tan\theta\right) = 4.42 \times 10^5\ \mathrm{V/m}$$

Problem 2.2 (SPhO 2015)**.** A particle of mass m and charge q, travelling with initial velocity v from far away, undergoes a head-on "collision" with an identical particle initially at rest. (i) Find the distance of closest approach. (ii) Find the velocities at closest approach. (iii) Find the final velocities.

Solution. Treat our system as both particles. We can first write the COE equation:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{kq^2}{r}$$

No net external force acts on the system, so we can also write our COM equation:

$$mv = mv_1 + mv_2 \quad \Rightarrow \quad v = v_1 + v_2$$

At closest approach, the particles should move with no relative velocity, thus

$$v_1 = v_2$$

These three equations are enough to solve for our desired variables, v_1, v_2, r at closest approach.

(i) Solving the three equations, you should obtain

$$r = \frac{4kq^2}{mv^2}$$

(ii) You should also obtain

$$v_1 = v_2 = \frac{v}{2}$$

(iii) By invoking symmetry, the final state of the system is similar to the initial state, except the particles have swapped roles. Hence,

$$v_{1,\infty} = 0, \ v_{2,\infty} = v$$

This must be correct as it satisfies COE and COM.

Problem 2.3. (Griffiths 2.18) In this problem you will derive some important results.
(a) Consider a sphere of radius R and uniform charge density ρ. Find the electric field everywhere.
(b) Now two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, are placed so that they partially overlap. Call the vector from the positive center to the negative center $\mathbf{d}$. Find the electric field in the overlap region.
(c) Now instead of a sphere, consider two infinitely long, uniformly charged cylinders that are partially overlapped and carrying uniform charge densities ρ and $-\rho$. Again, call the vector from the positive center to the negative center $\mathbf{d}$. Find the electric field within the overlap region.

Solution. **(a)** This part is same as the example in the notes above. Outside the sphere, the field is akin to that of a point charge. Within the sphere, the electric field is:

$$\vec{E}_{\text{in}} = \frac{\rho\vec{r}}{3\epsilon_0}$$

(b) Let $\vec{r}$ be the position vector of a point in the overlap region. We can use the principle of superposition. The electric field from the positive sphere, $\vec{E}_+$, at position $\vec{r}$ is given by our result for the field inside a sphere:

$$\vec{E}_+ = \frac{\rho\vec{r}}{3\epsilon_0}$$

The position vector of the point $\vec{r}$ relative to the center of the negative sphere is $\vec{r}' = \vec{r} - \vec{d}$. The electric field from the negative sphere, $\vec{E}_-$, at position $\vec{r}$ is:

$$\vec{E}_- = \frac{(-\rho)\vec{r}'}{3\epsilon_0} = \frac{-\rho(\vec{r} - \vec{d})}{3\epsilon_0}$$

The total electric field is the sum of the two:

$$\begin{aligned}\vec{E}_{\text{overlap}} &= \vec{E}_{+} + \vec{E}_{-} \\ &= \frac{\rho\vec{r}}{3\epsilon_0} + \frac{-\rho(\vec{r}-\vec{d})}{3\epsilon_0} \\ &= \frac{\rho\vec{d}}{3\epsilon_0}\end{aligned}$$

So, the electric field in the overlap region is *uniform.*

(c) First, we find the electric field for a single infinitely long cylinder of radius R and uniform charge density ρ. By cylindrical symmetry, the field must point radially away from the axis, $\vec{E} = E(s)\hat{s}$, where s is the radial distance from the axis. We use a cylindrical Gaussian surface of radius s and length L, coaxial with the charged cylinder.

For a point inside the cylinder ($s < R$): The enclosed charge is $Q_{\text{enc}} = \rho V_{\text{enc}} = \rho(\pi s^2 L)$. Applying Gauss's Law (the flux through the end caps is zero):

$$\oint \vec{E} \cdot d\vec{A} = E(s)(2\pi s L) = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\rho\pi s^2 L}{\epsilon_0}$$

Solving for $E(s)$:

$$E(s) = \frac{\rho s}{2\epsilon_0}$$

In vector form, with $\vec{s}$ being the radial vector perpendicular to the axis:

$$\vec{E}_{\text{in}} = \frac{\rho\vec{s}}{2\epsilon_0}$$

Now, consider two overlapping cylinders. The positive cylinder (density ρ) has its axis at the origin, and the negative cylinder (density $-\rho$) has its axis displaced by a vector $\vec{d}$. Let $\vec{s}$ be the position vector of a point in the overlap region from the axis of the positive cylinder. The field from the positive cylinder is:

$$\vec{E}_{+} = \frac{\rho\vec{s}}{2\epsilon_0}$$

The position vector from the axis of the negative cylinder is $\vec{s}' = \vec{s} - \vec{d}$. The field from the negative cylinder is:

$$\vec{E}_{-} = \frac{(-\rho)\vec{s}'}{2\epsilon_0} = \frac{-\rho(\vec{s}-\vec{d})}{2\epsilon_0}$$

The total field in the overlap region is:

$$\begin{aligned}\vec{E}_{\text{overlap}} &= \vec{E}_{+} + \vec{E}_{-} \\ &= \frac{\rho\vec{s}}{2\epsilon_0} + \frac{-\rho(\vec{s}-\vec{d})}{2\epsilon_0} \\ &= \frac{\rho\vec{d}}{2\epsilon_0}\end{aligned}$$

The electric field within the overlapping region of the cylinders is also *uniform.*

Problem 2.4. A soap film is made of a conductive liquid with uniform surface charge σ. Given the surrounding pressure is P_o, what is the pressure within the film? Neglect surface tension.

Solution. A soap film is a thin sheet of liquid with two surfaces. Since the liquid is conductive, the charge resides on these surfaces. The problem states a uniform surface charge density σ. We interpret this as the total charge per unit area of the film. By symmetry, this charge is distributed equally on the two surfaces. Let's denote the charge density on a single surface as σ'.

$$\sigma' = \frac{\sigma}{2}$$

Now, consider the electrostatic force exerted by one surface on the other. The electric field produced by a single, large, flat sheet of charge with density σ' is given by:

$$E' = \frac{\sigma'}{2\epsilon_0}$$

This field is uniform and directed perpendicularly away from the sheet.

Each surface of the soap film experiences a force due to the electric field created by the other surface. Let's calculate the force per unit area (which is pressure) on one surface. This electrostatic pressure, P_{elec}, is the charge density of that surface multiplied by the electric field from the other surface.

$$P_{\text{elec}} = \sigma' \cdot E' = \left(\frac{\sigma}{2}\right) \cdot \frac{(\sigma/2)}{2\epsilon_0} = \left(\frac{\sigma}{2}\right) \cdot \left(\frac{\sigma}{4\epsilon_0}\right)$$

$$P_{\text{elec}} = \frac{\sigma^2}{8\epsilon_0}$$

This is a repulsive pressure, so it acts outwards on each surface, pushing them apart. We get the pressure balance equation:

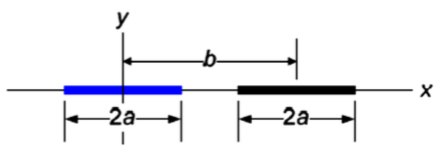
$$P_o = P + P_{\text{elec}}$$

Substituting the expression we found for P_{elec}:

$$P = P_o - \frac{\sigma^2}{8\epsilon_0}$$

The pressure inside the film is lower than the surrounding pressure because the mutual electrostatic repulsion of the two charged surfaces helps to counteract the external pressure.

Problem 2.5. Two identical thin rods of length $2a$ carry equal charges $+Q$ uniformly distributed along their lengths. The rods lie along the x-axis with their centres separated by a distance $b > 2a$.



(a) Show that the magnitude of the force exerted by the left rod on the right one is

$$F = \frac{Q^2}{16\pi\varepsilon_0\, a^2} \ln\left(\frac{b^2}{b^2 - 4a^2}\right).$$

(b) Explain qualitatively how this force would be different if each rod were a conductor.

Solution. **(a)** We will calculate the force by first finding the electric field of the left rod and then integrating the force this field exerts on the right rod. The linear charge density on each rod is $\lambda = Q/(2a)$.

Let's place the origin at the center of the left rod, so it extends from $x' = -a$ to $x' = a$. The electric field dE at a point x on the axis (for $x > a$) from an infinitesimal charge element $dq = \lambda dx'$ at position x' is:

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx'}{(x-x')^2}$$

The total field $E(x)$ is the integral over the length of the rod:

$$\begin{aligned}
E(x) &= \int_{-a}^{a} \frac{\lambda}{4\pi\epsilon_0}\frac{dx'}{(x-x')^2} \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{x-x'}\right]_{-a}^{a} \\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{x-a} - \frac{1}{x+a}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{\lambda(2a)}{4\pi\epsilon_0(x^2-a^2)}
\end{aligned}$$

Since $\lambda = Q/2a$, this simplifies to:

$$E(x) = \frac{Q}{4\pi\epsilon_0(x^2-a^2)}$$

The right rod extends from $x = b - a$ to $x = b + a$. The force dF on an infinitesimal charge element $dq' = \lambda dx$ at position x on the right rod is:

$$dF = dq'E(x) = (\lambda dx)\frac{Q}{4\pi\epsilon_0(x^2-a^2)} = \frac{Q^2}{8\pi\epsilon_0 a(x^2-a^2)}dx$$

To find the total force F, we integrate over the length of the right rod:

$$F = \int_{b-a}^{b+a} \frac{Q^2}{8\pi\epsilon_0 a}\frac{1}{x^2-a^2}dx$$

Using the standard integral $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$:

$$\begin{aligned}
F &= \frac{Q^2}{8\pi\epsilon_0 a}\left[\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|\right]_{b-a}^{b+a} \\
&= \frac{Q^2}{16\pi\epsilon_0 a^2}\left(\ln\left|\frac{(b+a)-a}{(b+a)+a}\right| - \ln\left|\frac{(b-a)-a}{(b-a)+a}\right|\right) \\
&= \frac{Q^2}{16\pi\epsilon_0 a^2}\left(\ln\left(\frac{b}{b+2a}\right) - \ln\left(\frac{b-2a}{b}\right)\right) \\
&= \frac{Q^2}{16\pi\epsilon_0 a^2}\ln\left(\frac{b/(b+2a)}{(b-2a)/b}\right) \\
&= \frac{Q^2}{16\pi\epsilon_0 a^2}\ln\left(\frac{b^2}{(b+2a)(b-2a)}\right) \\
&= \frac{Q^2}{16\pi\epsilon_0 a^2}\ln\left(\frac{b^2}{b^2-4a^2}\right)
\end{aligned}$$

This completes the proof.

(b) Explain qualitatively how this force would be different if each rod were a conductor.

If the rods were conductors, the charges ($+Q$ on each) would be free to move. Due to the repulsive electrostatic force between the like charges on the two rods, the charges on each rod would redistribute themselves to be as far as possible from the other rod.

- On the left rod, the charge density would become non-uniform, with more charge accumulating on the far left end (at $x = -a$) and less charge on the right end (at $x = a$).
- Similarly, on the right rod, charge would be pushed away from the left rod, resulting in a higher charge density on the far right end (at $x = b + a$) and a lower charge density on the left end (at $x = b - a$).

This charge redistribution means that the effective "center of charge" for each rod moves farther away from the other. Since the electrostatic force is inversely proportional to the square of the distance between charges, increasing the average distance between the interacting charges will **decrease** the magnitude of the repulsive force.

Therefore, the force between two conducting rods would be **smaller** than the force between two uniformly charged (insulating) rods.

Problem 2.6. A point charge e is placed at a distance R from the centre of a metallic sphere of radius a, with $R > a$. The sphere is conducting, but insulated from the surroundings and is electrically neutral.
(a) Determine the potential on the surface of the sphere.
(b) Determine the resultant force acting on the charge.

Solution. This problem is a textbook setup of method of images for a sphere.

(a) The electric field outside the sphere is equivalent to the field generated by the external charge e plus two image charges placed inside the sphere. An image charge $q' = -e\frac{a}{R}$ is placed at a distance $b = \frac{a^2}{R}$ from the center, on the line connecting the center to e. This charge, in conjunction with e, creates a potential of zero on the spherical surface $r = a$. A second image charge, $q'' = +e\frac{a}{R}$, is placed at the center of the sphere ($r = 0$). This is necessary to ensure the total charge of the image system is $q' + q'' = 0$, which correctly models the electrically neutral sphere. Also, the potential on the surface of the conductor is constant. We can calculate it by summing the potentials from the three charges (the external charge e, and the image charges q' and q'').

$$V_{\text{surface}} = V_e + V_{q'} + V_{q''}$$

By construction, the potential on the surface $r = a$ due to the charge e and the image charge q' is zero. Thus, the potential on the sphere is determined solely by the image charge q'' at the center.

$$V_{\text{surface}} = V_{q''} = \frac{1}{4\pi\epsilon_0}\frac{q''}{a}$$

Substituting the value of q'':

$$V_{\text{surface}} = \frac{1}{4\pi\epsilon_0}\frac{(ea/R)}{a} = \frac{e}{4\pi\epsilon_0 R}$$

(b) The force on the point charge e is the vector sum of the forces exerted by the two image charges, q' and q''. Both forces act along the line connecting the center, the image charges, and the external charge.

The force between e and the central charge $q'' = +ea/R$ is repulsive. The distance is R.

$$F_{e,q''} = \frac{1}{4\pi\epsilon_0}\frac{e \cdot q''}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{e(ea/R)}{R^2} = \frac{e^2 a}{4\pi\epsilon_0 R^3}$$

The force between e and the image charge $q' = -ea/R$ at distance $b = a^2/R$ is attractive. The distance between them is $d = R - b = R - \frac{a^2}{R} = \frac{R^2-a^2}{R}$.

$$F_{e,q'} = \frac{1}{4\pi\epsilon_0}\frac{e\cdot q'}{d^2} = \frac{1}{4\pi\epsilon_0}\frac{e(-ea/R)}{\left(\frac{R^2-a^2}{R}\right)^2} = -\frac{1}{4\pi\epsilon_0}\frac{e^2aR}{(R^2-a^2)^2}$$

The total force is the sum of these two components.

$$\begin{aligned}
F_{\text{total}} &= F_{e,q''} + F_{e,q'} \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{e^2a}{R^3} - \frac{e^2aR}{(R^2-a^2)^2}\right) \\
&= \frac{e^2a}{4\pi\epsilon_0}\left(\frac{(R^2-a^2)^2 - R\cdot R^3}{R^3(R^2-a^2)^2}\right) \\
&= \frac{e^2a}{4\pi\epsilon_0}\left(\frac{(R^4-2a^2R^2+a^4) - R^4}{R^3(R^2-a^2)^2}\right) \\
&= \frac{e^2a}{4\pi\epsilon_0}\left(\frac{a^4-2a^2R^2}{R^3(R^2-a^2)^2}\right) \\
&= -\frac{e^2a^3(2R^2-a^2)}{4\pi\epsilon_0R^3(R^2-a^2)^2}
\end{aligned}$$

The negative sign indicates that the net force on the charge e is attractive, directed towards the center of the sphere.

Problem 2.7. A circular ring of radius R_1 carries a uniformly distributed charge Q. It is placed near a neutral sphere of radius R_2, with their centres separated by a distance L. The plane of the ring is perpendicular to the straight line connecting the centres of the ring and the sphere. Determine the average electric potential on the surface of the sphere, $\overline{U}$.

Solution. There are various ways to do this problem, one of the ways appear in Griffiths as an idea. Here we will present a slicker method. In general, the total potential energy of such a system has three contributions: interactions between charges in the ring itself, interactions between charges in the sphere itself, and the ring-sphere interactions. We focus on the last contribution, from which we can write:

$$QV = q\bar{U}$$

where q is the total charge on the sphere, and V is the average potential on the ring due to the sphere. Notice that our sphere being neutral ($q = 0$) does not change the average potential on the sphere due to the ring. Thus, we work in the hypothetical assumption that q is nonzero. In such a scenario, we have a sphere of radius R_2 and total charge q. All points on the ring of radius R_1 are located at a constant distance d from the center of the sphere, which is at a distance L along the axis. By the Pythagorean theorem:

$$d = \sqrt{L^2 + R_1^2}$$

Therefore, by the Shell Theorem, the potential created by the charged sphere is constant everywhere on the ring:

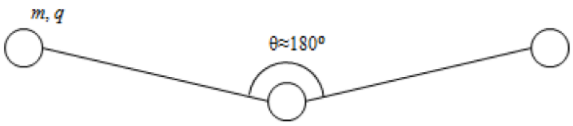
$$V = \frac{q}{4\pi\epsilon_0\sqrt{L^2+R_1^2}}$$

$$Q\left(\frac{q}{4\pi\epsilon_0\sqrt{L^2+R_1^2}}\right) = q\bar{U}$$

The arbitrary charge q cancels from both sides, leaving us with the desired average potential $\bar{U}$:

$$\bar{U} = \frac{Q}{4\pi\epsilon_0\sqrt{L^2 + R_1^2}}$$

Problem 2.8 (SPhO 2015)**.** 3 identical coins of mass m and charge q are connected by 2 strings of length d. Given that the period of oscillation is T, find q.



Solution. Let θ be defined as per the diagram. We define the deviation from equilibrium, $\phi = \theta - \pi, \ |\phi| << 1$. Label the coins $1, 2, 3$ from left to right.

The strings are assumed to be inextensible, thus the EPE between coins 1 and 2 and between coins 2 and 3 is constant (and doesn't vary with θ). Only the EPE between coins 1 and 3 varies with θ:

$$U_{1,3} = \frac{kq^2}{2d\sin\left(\frac{\theta}{2}\right)} = \frac{kq^2}{2d\sin\left(\frac{\pi}{2} + \frac{\phi}{2}\right)} = \frac{kq^2}{2d\cos\left(\frac{\phi}{2}\right)} \approx \frac{kq^2}{2d\left(1 - \frac{\phi^2}{8}\right)} \approx \frac{kq^2}{2d}\left(1 + \frac{\phi^2}{8}\right)$$

The constant EPE terms don't matter. Evaluating $\frac{d^2U}{d\phi^2}$ at the equilibrium position, we can find the effective spring constant:

$$k_{eff} = \left.\frac{d^2U}{d\phi^2}\right|_{\phi=0} = \frac{kq^2}{8d}$$

Now, we can evaluate the KE in terms of $\dot{\phi}$. To first order, we assume coin 2 is roughly stationary, and the "vertical" speed of coins 1 and 3 are

$$v_1 = v_3 = \frac{\dot{\phi}d}{2}$$

Hence, the total KE is

$$K = 2\left(\frac{1}{2}m\left(\frac{\dot{\phi}d}{2}\right)^2\right) = \frac{1}{4}md^2\dot{\phi}^2 = \frac{1}{2}m_{eff}\dot{\phi}^2$$

implying that the effective mass is

$$m_{eff} = \frac{1}{2}md^2$$

We can find the angular frequency and period of oscillation using k_{eff} and m_{eff}:

$$\omega = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{kq^2}{4md^3}} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4md^3}{kq^2}}$$

We can rearrange to find q as desired:

$$q = \frac{4\pi}{T}\sqrt{\frac{md^3}{k}}$$

Problem 2.9. A cube of length l possesses a uniform volume charge density ρ. Find the ratio of the electric potential at one of its vertices to that at its centre. *Hint: use scaling arguments.*

Solution. Observe that the electric potential is proportional to charge and inversely proportional to a length dimension. The potential at the center of a cube of side length $2l$ is then 4 times the potential at the center of a cube of side length l as its volume and thus charge are larger by a factor of 8 while its length is augmented by a factor of 2. Furthermore, the potential at the center of a cube of side length $2l$ is equal to 8 times the potential at the corner of a cube of side length l by the principle of superposition (as the large cube is made up of 8 smaller cubes). Equating these expressions for the potential at the center of a cube of side length $2l$, one can see that the ratio of the potential at the center of a cube of length l to that at the corner is $2:1$.

Problem 2.10. (Ricardo) A hole of radius R is carved out of a thin infinite plane with a positive surface charge density σ that is uniform. A point charge q of mass m is placed at the center of the hole. Neglecting all gravitational effects:
(a) Show that the center of the hole corresponds to an equilibrium position for the charge.
(b) Determine the stability of this equilibrium when the charge is displaced slightly in the direction normal to the plane. Discuss how the stability depends on the sign and magnitude of q.
(c) If the equilibrium is stable, find the angular frequency of small oscillations of the charge. You may find the result of the previous problem useful.

Solution. We have proven that the electric field due to an infinite plane of charge density σ is $\frac{\sigma}{2\epsilon_0}$ everywhere, directed outwards from the plane, and that the electric field at a height h above the center of a circular disk of charge density σ is $\frac{\sigma}{2\epsilon_0} - \frac{\sigma h}{2\sqrt{h^2+R^2}}$, directed outwards from the disk. Therefore, the electric field at a height h above the center of the hole in the infinite plane is

$$E(h) = \frac{\sigma}{2\epsilon_0} - \left(\frac{\sigma}{2\epsilon_0} - \frac{\sigma h}{2\sqrt{h^2+R^2}}\right) = \frac{\sigma h}{2\sqrt{h^2+R^2}}$$

and is directed normally outwards from the center of the hole. When $h = 0$, $E(0) = 0$ which shows that the center of the hole corresponds to an equilibrium position. Furthermore, since $E(h)$ is directed normally outwards, the equilibrium for $q > 0$ is unstable (as the Coulomb force tends to push it further away from the hole) while the equilibrium for $q < 0$ is stable (as the Coulomb force tends to correct its deviation from the hole). For small values of $h \ll R$, the electric field becomes

$$E(h) = \frac{\sigma h}{2R}.$$

Therefore, Newton's law yields, for the charge $q < 0$,

$$\begin{aligned} m\ddot{h} &= \frac{q\sigma h}{2R} \\ \implies \ddot{h} &= -\frac{-q\sigma}{2Rm}h, \end{aligned}$$

where $-q$ is a positive quantity. The above equation of motion describes a simple harmonic motion with angular frequency

$$\omega = \sqrt{\frac{-q\sigma}{2Rm}}.$$

Problem 2.11. A point charge q is placed a distance $a/2$ above the centre of a square of surface charge density σ and side length a. Find the force exerted by the square on the point charge. *Hint: the value $\frac{a}{2}$ was chosen for a special reason.*

Solution. By Newton's 3rd Law, the problem is equivalent to finding the force of the point charge on the square. Set up coordinates so that the square is in the xy plane, and its center is the origin. Then we have

$$\mathbf{F} = \sigma \int \mathbf{E} dS$$

where the surface integral is over the square. On the other hand, we know that $\mathbf{F}$ is along the $\hat{\mathbf{z}}$ direction by symmetry, so

$$F = \mathbf{F} \cdot \hat{\mathbf{z}} = \sigma \int E_z dS.$$

Now, since $d\mathbf{S}$ is parallel to $\hat{\mathbf{z}}$, this is in fact the same thing as

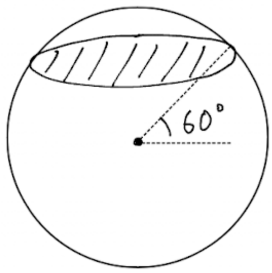
$$F = \sigma \int \mathbf{E} \cdot d\mathbf{S}$$

where the integral is just the electric flux through the square! By symmetry, this flux is $q/6\epsilon_0$, so

$$F = \frac{\sigma q}{6\epsilon_0}.$$

This specific problem relies on a one-time-trick, but using symmetry and shapes to determine flux is relatively common.

Problem 2.12. For a solid sphere of radius R and uniform charge density ρ, calculate:
(a) The electric field within the sphere.
(b) The electric flux through the circular area (see diagram below).



Solution. **(a)** The electric field within the sphere has been derived multiple times:

$$\vec{E}(\vec{r}) = \frac{\rho \vec{r}}{3\epsilon_0}$$

(b) A common mistake is to assume that the flux through the circular area is also the flux through the spherical cap. However, this is erroneous because there is enclosed charge. Thus, we solve this by constructing a closed Gaussian surface from the flat circular disk and the spherical cap it defines. Let this closed surface be S. Gauss's Law states that the net flux through S is the sum of the flux through the disk and the cap:

$$\Phi_{\text{net}} = \Phi_{\text{disk}} + \Phi_{\text{cap}} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

We will calculate Q_{enc} and Φ_{cap} and then solve for Φ_{disk}.

The enclosed charge is the density ρ times the volume of the spherical cap, V_{cap}. This volume is the volume of the spherical sector minus the volume of the cone.

- The cone's height is $H = R\cos(60°) = R/2$. Its base radius is $r = R\sin(60°) = R\sqrt{3}/2$. Its volume is $V_{\text{cone}} = \frac{1}{3}\pi r^2 H = \frac{\pi R^3}{8}$.

- The spherical sector's volume is $V_{\text{sector}} = \frac{2}{3}\pi R^2 h$, where the cap height is $h = R - H = R/2$. Thus, $V_{\text{sector}} = \frac{\pi R^3}{3}$.
- The enclosed volume is $V_{\text{cap}} = V_{\text{sector}} - V_{\text{cone}} = \frac{\pi R^3}{3} - \frac{\pi R^3}{8} = \frac{5\pi R^3}{24}$.

The enclosed charge is $Q_{\text{enc}} = \rho V_{\text{cap}} = \frac{5\pi\rho R^3}{24}$. We integrate $\vec{E} \cdot d\vec{A}$ over the curved cap surface. Here, $\vec{E} = \frac{\rho R}{3\epsilon_0}\hat{r}$ and $d\vec{A} = R^2 \sin\theta d\theta d\phi \hat{r}$.

$$\begin{aligned}\Phi_{\text{cap}} &= \int_0^{2\pi}\int_0^{\pi/3}\left(\frac{\rho R}{3\epsilon_0}\right)(R^2\sin\theta)d\theta d\phi \\ &= \frac{2\pi\rho R^3}{3\epsilon_0}[-\cos\theta]_0^{\pi/3} = \frac{2\pi\rho R^3}{3\epsilon_0}(1-\cos(\pi/3)) = \frac{2\pi\rho R^3}{3\epsilon_0}\left(\frac{1}{2}\right) = \frac{\pi\rho R^3}{3\epsilon_0}.\end{aligned}$$

From Gauss's Law, $\Phi_{\text{disk}} = Q_{\text{enc}}/\epsilon_0 - \Phi_{\text{cap}}$.

$$\Phi_{\text{disk}} = \frac{5\pi\rho R^3}{24\epsilon_0} - \frac{\pi\rho R^3}{3\epsilon_0} = \frac{5\pi\rho R^3 - 8\pi\rho R^3}{24\epsilon_0} = -\frac{3\pi\rho R^3}{24\epsilon_0} = -\frac{\pi\rho R^3}{8\epsilon_0}$$

where the negative sign denotes flux entering our Gaussian surface. We take the magnitude:

$$\Phi_E = \frac{\pi\rho R^3}{8\epsilon_0}$$

Problem 2.13. A particle of charge q is moved from infinity to the centre of a hollow conducting spherical shell of inner radius R and thickness t, through a very tiny hole in the shell. Determine the work required.

Solution. The ideal way to solve this problem is by using electric field energy. The work required, W, to move the charge q from infinity to the center of the shell is equal to the change in the total electrostatic energy of the system: $W = \Delta U = U_{\text{final}} - U_{\text{initial}}$. We calculate the energy by integrating the energy density, $u = \frac{1}{2}\epsilon_0 E^2$, over all space. To handle the infinite self-energy of the point charge, we exclude a small sphere of radius ϵ around the charge from our integration; this term will cancel out in the final subtraction.

Initially, the charge q is at infinity. The electric field is $E = \frac{q}{4\pi\epsilon_0 r^2}$. The initial energy stored in the field (outside the radius ϵ) is:

$$\begin{aligned}U_{\text{initial}} &= \frac{\epsilon_0}{2}\int_\epsilon^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 (4\pi r^2\, dr) = \frac{q^2}{8\pi\epsilon_0}\int_\epsilon^\infty \frac{1}{r^2}\, dr \\ &= \frac{q^2}{8\pi\epsilon_0}\left[-\frac{1}{r}\right]_\epsilon^\infty = \frac{q^2}{8\pi\epsilon_0\epsilon}\end{aligned}$$

In the final configuration, the charge q is at the center. This induces a charge $-q$ on the inner surface ($r = R$) and $+q$ on the outer surface ($r = R + t$). The field is non-zero in two regions:

- Inside the cavity ($r < R$): $E_{\text{in}} = \frac{q}{4\pi\epsilon_0 r^2}$
- Outside the shell ($r > R + t$): $E_{\text{out}} = \frac{q}{4\pi\epsilon_0 r^2}$

The field is zero inside the conductor ($R < r < R + t$). The final energy is the integral over these two regions:

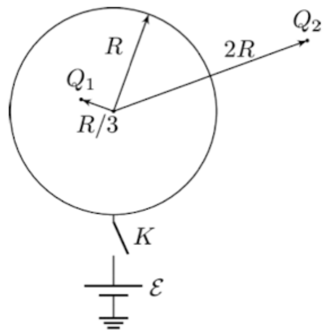
$$\begin{aligned}
U_{\text{final}} &= \frac{\epsilon_0}{2}\left[\int_{\epsilon}^{R} E_{\text{in}}^2 (4\pi r^2\,dr) + \int_{R+t}^{\infty} E_{\text{out}}^2 (4\pi r^2\,dr)\right] \\
&= \frac{q^2}{8\pi\epsilon_0}\left[\int_{\epsilon}^{R} \frac{1}{r^2}\,dr + \int_{R+t}^{\infty} \frac{1}{r^2}\,dr\right] \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\left[-\frac{1}{r}\right]_{\epsilon}^{R} + \left[-\frac{1}{r}\right]_{R+t}^{\infty}\right) \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\left(-\frac{1}{R} + \frac{1}{\epsilon}\right) + \left(0 + \frac{1}{R+t}\right)\right) = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{\epsilon} - \frac{1}{R} + \frac{1}{R+t}\right)
\end{aligned}$$

The work done by an external agent is $W = U_{\text{final}} - U_{\text{initial}}$:

$$\begin{aligned}
W &= \left[\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{\epsilon} - \frac{1}{R} + \frac{1}{R+t}\right)\right] - \left[\frac{q^2}{8\pi\epsilon_0\epsilon}\right] \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{R+t} - \frac{1}{R}\right) = -\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R+t}\right)
\end{aligned}$$

The negative sign indicates that the external agent does negative work, as the charge is attracted to the conducting shell.

Problem 2.14. An initially uncharged conducting spherical shell of radius R is connected to ground through a battery with EMF ε and an open switch K. A point charge Q_1 is placed inside the shell at a distance $R/3$ from its centre, and a point charge Q_2 is placed outside the shell at a distance $2R$ from its centre.



(a) Determine the potential at the centre of the shell when the switch K is open.
(b) The switch K is then closed and equilibrium is reached. Determine the charge on the shell.

Solution. **(a)** This problem focuses explicitly on properties of conductors and electrostatic shielding. When the switch is open, the conducting shell is isolated and electrically neutral (total charge is zero). A charge Q_1 inside the cavity induces a charge $-Q_1$ on the inner surface and $+Q_1$ on the outer surface. The potential at the center V_c is found by superposition.

The potential at the center due to a spherical shell of charge Q_s and radius R_s is $\frac{Q_s}{4\pi\epsilon_0 R_s}$.

- Potential from point charge Q_1 at distance $R/3$: $V_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R/3} = \frac{3Q_1}{4\pi\epsilon_0 R}$.
- Potential from point charge Q_2 at distance $2R$: $V_2 = \frac{1}{4\pi\epsilon_0}\frac{Q_2}{2R}$.
- Potential from inner induced charge ($-Q_1$ at R): $V_{\text{inner}} = \frac{1}{4\pi\epsilon_0}\frac{-Q_1}{R}$.
- Potential from outer induced charge ($+Q_1$ at R): $V_{\text{outer}} = \frac{1}{4\pi\epsilon_0}\frac{+Q_1}{R}$.

The total potential is the sum:

$$\begin{aligned} V_c &= V_1 + V_2 + V_{\text{inner}} + V_{\text{outer}} \\ &= \frac{3Q_1}{4\pi\epsilon_0 R} + \frac{Q_2}{8\pi\epsilon_0 R} - \frac{Q_1}{4\pi\epsilon_0 R} + \frac{Q_1}{4\pi\epsilon_0 R} \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{3Q_1}{R} + \frac{Q_2}{2R}\right) \end{aligned}$$

(b) When the switch is closed, the shell is held at a potential $V_{\text{shell}} = \varepsilon$. Let the total charge on the shell be Q_{shell}. To maintain $E = 0$ inside the conductor, the charge on the inner surface must be $q_{\text{inner}} = -Q_1$. The charge on the outer surface is therefore $q_{\text{outer}} = Q_{\text{shell}} - q_{\text{inner}} = Q_{\text{shell}} + Q_1$.

The potential at the shell is the sum of potentials from four distinct contributions: $Q_1, q_{inner}, Q_2, q_{outer}$. We deal with the cases in pairs. Firstly, if only Q_1 and q_{inner} were present, there would be no electric field outside the sphere due to shielding, implying that the potential at the surface of the sphere is zero. Next, if only Q_2 and q_{outer} were present, the electric field within the sphere is zero also due to shielding. Therefore, to calculate the potential at the surface due to all four contributions, it suffices to calculate the potential at the centre due to Q_2 and q_{outer}, since Q_1 and q_{inner} don't contribute to the potential at the surface, and Q_2 and q_{outer} produce no potential difference between the centre and the surface of the sphere.

- Potential at the centre due to q_{outer}: $V_{\text{outer}} = \frac{q_{\text{outer}}}{4\pi\epsilon_0 R} = \frac{Q_{\text{shell}}+Q_1}{4\pi\epsilon_0 R}$.
- Potential at centre due to Q_2: $V_2 = \frac{Q_2}{4\pi\epsilon_0(2R)} = \frac{Q_2}{8\pi\epsilon_0 R}$.

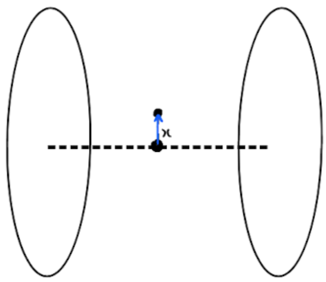
The total potential on the shell must be ε:

$$V_{\text{shell}} = V_{\text{outer}} + V_2 = \frac{Q_{\text{shell}} + Q_1}{4\pi\epsilon_0 R} + \frac{Q_2}{8\pi\epsilon_0 R} = \varepsilon$$

Rearranging gives:

$$Q_{\text{shell}} = 4\pi\epsilon_0 R\varepsilon - Q_1 - \frac{Q_2}{2}$$

Problem 2.15. Two circular plates of radius R are placed a distance d apart. They are charged with surface charge densities $-\sigma$. A small particle of charge $+q$ and mass m is placed at the midpoint of the line connecting the centres of the two plates. It is displaced slightly in a direction perpendicular to this line. Determine the period of small oscillations it performs.



Solution. This problem can be solved via direct integration of the potential function, then performing a derivative to determine the desired electric field. However, such a procedure is tedious and likely to yield careless mistakes.

Instead, we consider a more elegant solution using Gauss' Law. First, we define the electric field $E_{\text{disc}}(x)$ on the axis of a single disc of radius R and uniform surface charge density $-\sigma$, placed at the origin. For a point on the positive x-axis, the field is attractive (points in the $-x$ direction) and is given by:

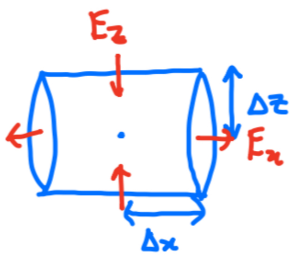
$$E_{\text{disc}}(x) = -\frac{\sigma}{2\epsilon_0}\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

The negative sign indicates the field direction is along the $-x$ axis for a positive charge σ. Since our charge is $-\sigma$, the expression correctly represents the field magnitude and direction.

We consider a small cylindrical Gaussian surface of radius z and height $2\Delta x$, centered at the origin, which is the equilibrium position of the charge $+q$. The cylinder's axis aligns with the x-axis. As there is no charge enclosed, the net electric flux through the cylinder's surface is zero according to Gauss's Law ($\oint \vec{E} \cdot d\vec{A} = 0$).

This means that the flux leaving through the curved radial side (Φ_{radial}) must be equal and opposite to the net flux through the two circular end caps (Φ_{ends}).

$$\Phi_{\text{ends}} + \Phi_{\text{radial}} = 0$$



The flux through the radial side, which has an area of $A_{\text{radial}} = (2\pi z)(2\Delta x)$, is due to the radial component of the electric field, E_z.

$$\Phi_{\text{radial}} = E_z \cdot A_{\text{radial}} = E_z(4\pi z \Delta x)$$

The net flux through the end circular faces depends on the change in the total axial field E_x between $x = +\Delta x$ and $x = -\Delta x$. For small Δx, this can be expressed using the derivative of the total field at the origin:

$$\Phi_{\text{ends}} \approx 2\left(2\Delta x \left.\frac{dE_x}{dx}\right|_{x=\frac{d}{2}}\right)(\pi z^2)$$

where the first factor of 2 accounts for both faces, and the second factor of 2 accounts for the fact that there are two plates. Equating the fluxes ($\Phi_{\text{radial}} = -\Phi_{\text{ends}}$) and solving for E_z gives:

$$E_z = -z \left.\frac{dE_x}{dx}\right|_{x=\frac{d}{2}}$$

To find the flux through the circular faces:

$$\frac{dE_{\text{x}}}{dx} = \frac{\sigma R^2}{2\epsilon_0(x^2+R^2)^{3/2}}$$

Now, substitute this back into the expression for the radial field E_z:

$$E_z = -\frac{z}{2}\left(\frac{\sigma R^2}{\epsilon_0(R^2+d^2/4)^{3/2}}\right)$$

The restoring force F_z on the particle of charge $+q$ is $F_z = qE_z$.

$$F_z = -\left[\frac{q\sigma R^2}{2\epsilon_0(R^2+d^2/4)^{3/2}}\right] z$$

This force is of the form $F = -kz$, which describes Simple Harmonic Motion. The "spring constant" k is:

$$k = \frac{q\sigma R^2}{2\epsilon_0(R^2+d^2/4)^{3/2}}$$

The period of oscillation T for a particle of mass m is given by $T = 2\pi\sqrt{m/k}$.

$$T = 2\pi\sqrt{\frac{m}{\frac{q\sigma R^2}{2\epsilon_0(R^2+d^2/4)^{3/2}}}}$$

Simplifying this expression yields the final answer:

$$T = 2\pi\sqrt{\frac{2m\epsilon_0(R^2+d^2/4)^{3/2}}{q\sigma R^2}}$$

3 Advanced Problems

Problem 3.1. This problem involves a cool idea that has appeared elsewhere before. There are two point charges, $q_1 > 0$ and $q_2 < 0$, in empty space. An electric field line leaves q_1 at an angle α from the line connecting the two charges. Determine whether this field line hits q_2, and if so, at what angle β from the line connecting the two charges. (Hint: this can be done without solving any differential equations.)

Solution. The cool idea here is that you can form a unorthodox Gaussian surface bounded by the field lines! Suppose the field line does hit q_2. Rotate the field line about the line connecting the two charges, forming our desired Gaussian surface. Because no electric field lines go across this surface, the total charge inside must be zero by Gauss's Law. Now, this surface envelops "slices" of each point charge. (If you're not happy with "slicing a point charge", just replace the point charges with tiny uniformly charged spheres; everything outside stays the same.) The solid angle of the first point charge enveloped is

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\alpha} \sin\theta \, d\theta = 2\pi(1 - \cos\alpha)$$

so the amount of charge enclosed is

$$\frac{\Omega}{4\pi} q_1 = \frac{1 - \cos\alpha}{2} q_1 = q_1 \sin^2 \frac{\alpha}{2}.$$

Reasoning similarly for the other surface, we have

$$q_1 \sin^2 \frac{\alpha}{2} = |q_2| \sin^2 \frac{\beta}{2}$$

and the field line hits q_2 if there is a solution for β, i.e. when $|q_1/q_2| \sin^2(\alpha/2) < 1$.

Problem 3.2 (200 Puzzling Physics Problems)**.** Consider a uniformly charged spherical shell of radius R and total charge Q. (i) Find the net electrostatic force that the Southern hemisphere exerts on the Northern hemisphere. (ii) Generalise part (i) to the case where the sphere is split into two parts by a plane whose minimum distance to the sphere's centre is h. (iii) Generalise part (i) to the case of two hemispherical shells with uniform charge density, opposite orientation, and the same centre, but have different total charges q and Q and different radii r and R respectively, where $r < R$.

Solution. (i) This part is easily solved using electrostatic pressure. The electric field is

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}$$

hence the electrostatic pressure is

$$P_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{Q^2}{32\pi^2\varepsilon_0 R^4}$$

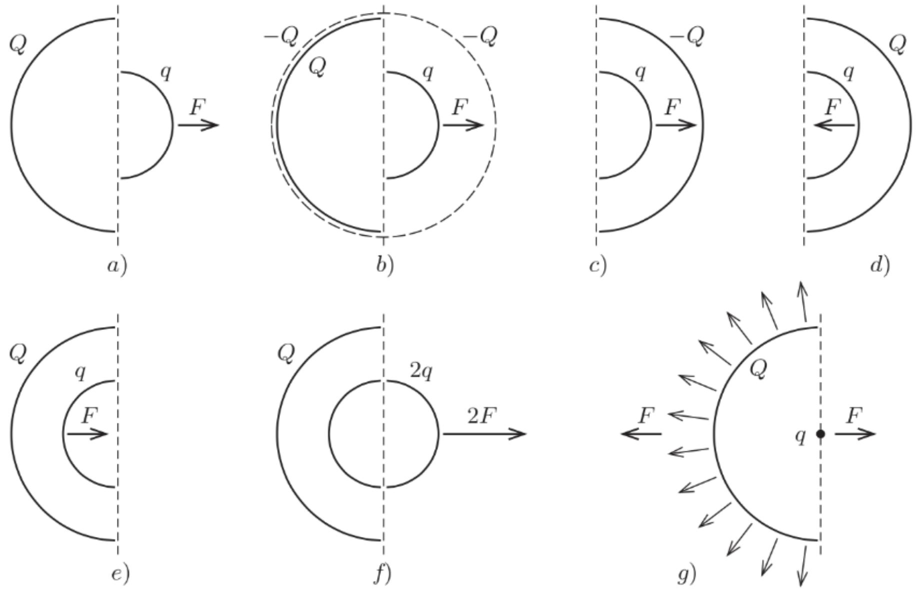
The face in contact has area πR^2, hence

$$F = P_E \pi R^2 = \frac{Q^2}{32\pi\varepsilon_0 R^2}$$

(ii) In this case, the area of the face is $\pi\left(R^2 - h^2\right)$, hence

$$F = P_E \pi \left(R^2 - h^2\right) = \frac{Q^2\left(R^2 - h^2\right)}{32\pi\varepsilon_0 R^4}$$

(iii) This is the difficult part, which requires you to make a *series* of ingenious observations to simplify the configuration.



The force of interest F is shown in (a). The reasoning for each step is detailed below:

1. In (b), we place a hollow spherical shell of total charge $-2Q$ and radius just slightly greater than R. The left side cancels out with Q, so we have $-Q$ left on the right side. By the shell theorem, this hollow shell will not affect F. This gives us (c).
2. Consider (c), but with the larger hemisphere being of charge Q instead of $-Q$. This simply flips the direction of F, giving us (d).
3. Now, reflect (d) about the dashed line, giving us (e). Note that the direction of F flips again due to the reflection.
4. Superpose the cases (a) and (e) together. The two inner hemispheres combine to give a hollow sphere of radius r and total charge $2q$, while the outer hemisphere remains of charge Q and radius R. As the forces F are in the same direction for both cases, they add up to give $2F$. This gives us (f).
5. By the shell theorem, the hemisphere of charge $2q$ experiences a force equivalent to having a point charge $2q$ at the centre. Since the force scales linearly with charge, if we divide the charge by 2, the force experienced by a point charge q at the centre of a hemisphere of charge Q and radius R will be F.

We have replaced our problem into something much easier to solve, since the point charge configuration gives the same force! To solve this, define the hemisphere's surface charge density as $\sigma = \frac{Q}{2\pi R^2}$. We can find the force on the hemisphere by the charge by integrating the field from the point charge over the area vectors of the hemisphere:

$$\mathbf{F} = \int \frac{q}{4\pi\varepsilon_0 R^2}\sigma \, d\mathbf{S} = \frac{q\sigma}{4\pi\varepsilon_0 R^2}\int d\mathbf{S}$$

Hence, the magnitude of the desired force is

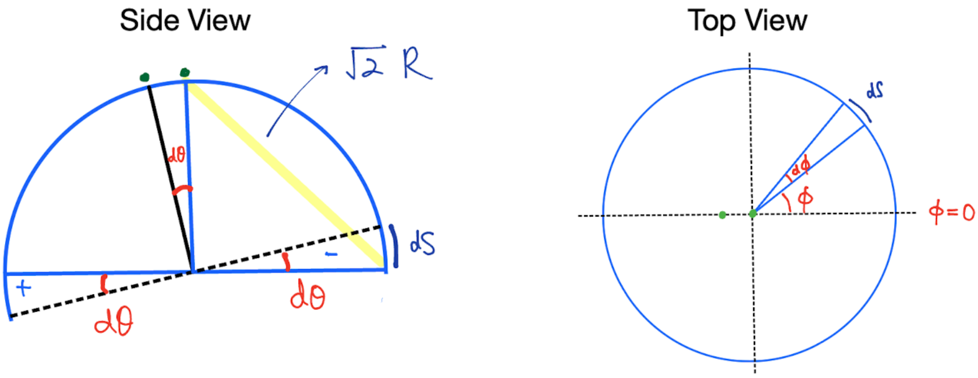
$$F = \frac{q\sigma}{4\pi\varepsilon_0 R^2}\left(\pi R^2\right) = \frac{q\sigma}{4\varepsilon_0} = \frac{Qq}{8\pi\varepsilon_0 R^2}$$

which is surprisingly independent of r!

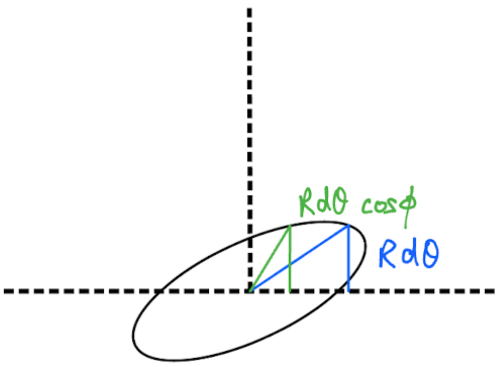
Problem 3.3. A thin-walled dielectric hemisphere, negatively charged with surface density $-\sigma$, is placed on a horizontal table. A point-like ball of mass m is carefully placed at its top. Determine the minimum positive charge Q of the ball such that it remains in a state of stable equilibrium at the top of the hemisphere. The charges on the hemisphere and the ball are not redistributed. The free-fall acceleration is g.

Solution. This problem is from IZhO 2022, but the original solution was erroneous, the correct version is presented here.

Let the ball be displaced from its original position by a small distance $Rd\theta$. To find the electrostatic force, we consider a superposition of the exact same hemisphere positioned symmetrically below the new displaced position, plus 2 new "slices" of charge with opposite polarity (see figure below). Due to symmetry, the force due to a complete hemisphere is normal to the surface and can thus be neglected here - we focus solely on the tangential forces due to our two "slices".



The 3D view is shown below. Most importantly, try to visualize how the height of each area element is approximately equal to $Rd\theta\cos(\phi)$.



To determine the charge element $dq = -\sigma dS$, we use $dS = R^2\cos(\phi)d\theta d\phi$.

To first order approximations, the electric field due to each charge element is approximately a distance $\sqrt{2}R$ from ball. We write Couloumb's Law:

$$d\vec{F} = \frac{Qdq}{4\pi\epsilon_0 r^3}\vec{r}.$$

Projecting this on the tangential direction:

$$dF_Q = \frac{kQdq}{(\sqrt{2}R)^2}\cos(\phi)\frac{R}{\sqrt{2}R}$$

Substituting our expression for dq yields:

$$dF_Q = -\frac{kQ\sigma}{2\sqrt{2}R}\cos^2(\phi)d\theta d\phi$$

The definite integral evaluates to $\int_{-\pi/2}^{\pi/2}\cos^2\phi\, d\phi = \frac{\pi}{2}$. Also, remember to multiply by 2 since there are two slices of opposite polarity that contribute the same amount of force (to first order). Simplifying yields the final expression for the force:

$$F_Q = -\frac{\sigma Q}{8\sqrt{2}\varepsilon_o}d\theta$$

The force of gravity when projected on the tangential direction is obtained as

$$F_g = mgd\theta.$$

The minimum charge of the ball is determined by the equality of forces for equilibrium:

$$F_g + F_Q = 0$$

which leads to the final answer:

$$Q = \frac{8\sqrt{2}\varepsilon_0 mg}{\sigma}$$

Obviously, for larger charges the equilibrium position is stable.